THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2020B Advanced Calculus II
Suggested Solutions for Homework 7

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1. Assuming that the field is conservative, find a potential function for the field and evaluate the integral

$$\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}dz.$$

Solution. Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ where $M = \frac{1}{y}, N = \left(\frac{1}{z} - \frac{x}{y^2}\right), P = -\frac{y}{z^2}$. Suppose $f(x,y,z)$ is a potential function for $\vec{F}$ where $\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N, \frac{\partial f}{\partial z} = P$. Integrating M with respect to x, we find that

$$f(x,y,z) = \int M dx = \int \frac{1}{y}dx = \frac{x}{y} + g(y,z) + C_1$$

where C_1 is a constant and g is a function of y and z. Then we find that

$$\left(\frac{1}{z} - \frac{x}{y^2}\right) = \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial z} = \frac{1}{z}.$$

Then integrating g with respect to y, we obtain

$$g(y,z) = \frac{y}{z} + h(z) + C_2$$

where C_2 is a constant and h is a function only in z. Hence $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + h(z) + C_2$. Then similarly, we find that

$$-\frac{y}{z^2} = \frac{\partial f}{\partial z} = -\frac{y}{z^2} + \frac{\partial h}{\partial z} \Rightarrow \frac{\partial h}{\partial z} = 0 \Rightarrow h(z) = C_3$$

where C_3 is a constant. Hence, we find that

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} + C$$

for some constant C. Using this potential function, we find that

$$\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}dz = f(2,2,2) - f(1,1,1) = \frac{2}{2} + \frac{2}{2} - \frac{1}{1} - \frac{1}{1} = 0.$$

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2. Find a potential function for $\vec{F}$ where

$$\vec{F} = \frac{2x}{y}\vec{i} + \left(\frac{1-x^2}{y^2}\right)\vec{j}, \quad \{(x,y) : y > 0\}.$$

Solution. Note that the domain is the upper half-plane and is simply connected. We write that $\vec{F} = M\vec{i} + N\vec{j}$ where $M = \dfrac{2x}{y}, N = \dfrac{1-x^2}{y^2}$ and we note that both M, N have continuous partial derivatives on the domain (since $y > 0$). We check that

$$\frac{\partial M}{\partial y} = -\frac{2x}{y^2} = \frac{\partial N}{\partial x}$$

and hence we see that $\vec{F}$ is indeed conservative.

Integrating, we have

$$f(x,y) = \int M dx = \int \frac{2x}{y} dx = \frac{x^2}{y} + g(y) + C_1$$

where C_1 is a constant and g is a function of y. We then have

$$\frac{1-x^2}{y^2} = \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y^2}.$$

Integrating g with respect to y, we then obtain

$$g(y) = \int \frac{1}{y^2} dy = -\frac{1}{y} + C_2$$

where C_2 is a constant. Hence, we find that

$$f(x,y) = \frac{x^2}{y} - \frac{1}{y} + C$$

where C is a constant, and $\vec{F} = \nabla\left(\dfrac{x^2-1}{y}\right)$. ◀

3. Let C be an oriented closed curve in a region $\Omega \subseteq \mathbb{R}^n$ given by $\gamma : [a,b] \to \Omega$. Recall that we say that C continuously deforms to a point, say Q, if there exists a continuous function $R : [a,b] \times [0,1] \to \Omega$ such that $R(t,0) = \gamma(t)$ and $R(t,1) = Q$ for every $t \in [a,b]$. Show that in $\Omega = \mathbb{R}^3 \setminus \{(0,0,0)\}$, any oriented closed curve C on the xy-plane away from the origin continuously deforms to a point.

Solution. Let C be an oriented closed curve in the xy-plane away from the origin given by the parameterization $\gamma : [a,b] \to \Omega$. Take $Q = (0,0,1)$ and define $R : [a,b] \times [0,1] \to \Omega$ such that

$$R(t,s) = (1-s)\gamma(t) + sQ.$$

Then at $s = 0$, we have that $R(t,0) = (1-0) \cdot \gamma(t) + 0 \cdot Q = \gamma(t)$ and at $s = 1, R(t,1) = (1-1) \cdot \gamma(t) + 1 \cdot Q = Q$. Clearly, R is continuous in t, s. It remains to check that $R(t,s)$ indeed lies in Ω for all $t \in [a,b]$ and $s \in [0,1]$. Note that when $s = 0$, $R(t,0) = \gamma(t) \in \Omega$ since $\gamma(t)$ is away from the origin by assumption. Also note that for any $s > 0$, the vector $R(t,s)$ has a non-zero z-component and hence is also away from the origin and lies in Ω. So we see that γ continuously deforms to $Q = (0,0,1)$ and we are done. ◀

4. Let C be the curve $\gamma : [0, 2\pi] \to \mathbb{R}^3$ given by $\gamma(\theta) = (\cos(\theta), \sin(\theta), \sin^2(\theta))$. Show that

$$\oint_C \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}} = 0.$$

(Hint: Problem 3.)

Solution. Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ where

$$M = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, N = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, P = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Then if we show $\vec{F}$ is conservative, we would be done since C is a closed curve. Problem 3 above shows that $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ is simply connected, and since $\vec{F}$ is defined on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$, we can apply the component test for conservative vector fields, that is, it is enough to show

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}.$$

We have

$$\frac{\partial M}{\partial y} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot \frac{1}{2} \cdot (x^2 + y^2 + z^2)^{-3/2} \cdot 2y = \frac{xy}{(x^2 + y^2 + z^2)^2} = \frac{\partial N}{\partial x}$$

and similarly,

$$\frac{\partial N}{\partial z} = \frac{yz}{(x^2 + y^2 + z^2)^2} = \frac{\partial P}{\partial y}, \frac{\partial M}{\partial z} = \frac{xz}{(x^2 + y^2 + z^2)^2} = \frac{\partial P}{\partial x}$$

as required. Hence, we conclude that $\vec{F}$ is conservative and we are done. ◀

5. (a) Find a potential function for the gravitational field

$$\vec{F} = -GmM \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

(G, m, and M are constants).

(b) Let P_1 and P_2 be points at distances s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM \left(\frac{1}{s^2} - \frac{1}{s^2} \right).$$

Solution. (a) Integrating $\dfrac{\partial f}{\partial x} = -GmM \dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}$, we have that

$$f(x, y, z) = \int -GmM \frac{x}{(x^2 + y^2 + z^2)^{3/2}} dx = \frac{GmM}{\sqrt{x^2 + y^2 + z^2}} + g(y, z) + C_1$$

where C_1 is a constant and g is a function of y and z. Then we have that

$$\frac{\partial f}{\partial y} = -GmM\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} = -GmM\frac{y}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0.$$

So $f(x,y,z) = \dfrac{GmM}{\sqrt{x^2+y^2+z^2}} + h(z) + C_2$ where C_2 is a constant and h is a function only of z. Then similarly we have that

$$\frac{\partial f}{\partial z} = -GmM\frac{z}{\sqrt{x^2+y^2+z^2}} + \frac{\partial h}{\partial z} = -GmM\frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial h}{\partial z} = 0.$$

Hence we conclude that

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}} + C$$

for some constant C.

(b) At $P_1 = (x_1, y_1, z_1)$, $s_1 = \sqrt{x_1^2+y_1^2+z_1^2}$ and at $P_2 = (x_2, y_2, z_2)$, $s_2 = \sqrt{x_2^2+y_2^2+z_2^2}$. Using the potential function found in part (a) above, we find that the work done in the gravitational field is given by

$$\int_{P_1}^{P_2} \vec{F} ds = f(s_2) - f(s_1) = GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$$

as required.

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